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Generalizations of the Tetrahedral and Octahedral Groups.

By G. A. Miller.

§ 1. *Introduction.*

A generalization of these groups is included in an article entitled "Generalization of the Groups of Genus Zero."[*] In the present paper many of the results of this article are greatly extended, and hence this paper may be regarded to be, in part, a generalization of a generalization of the tetrahedral and the octahedral groups. The fundamental importance of the groups of genus zero adds interest to these extensions, and the simple character of these extensions seems to justify the hope that they may find extensive applications. The present paper has close contact with a recent paper entitled "Finite Groups Which may be Defined by Two Operators Satisfying Two Conditions."[†]

In what follows, it is frequently necessary to speak of a group of order 24 which does not include any subgroup of order 12. As there is only one such group, it is completely determined by stating its order and the fact that it does not involve a subgroup of order 12. To secure greater brevity we shall hereafter refer to this group as the non-twelve G_{24}, or the non-twelve group of order 24. In accord with this notation, the tetrahedral group may be characterized as the non-six G_{12}, and the icosahedral group could be distinguished by the fact that it is the non-invariant-five group of order 60, since the icosahedral group is the only group of order 60 which involves more than one subgroup of order 5. While this method of notation frequently leads to brevity of statement without any material loss in clearness, it is not to be inferred that it solves all the difficulties connected with distinguishing the different groups of the same order.

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[*] *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 1.

[†] American Journal of Mathematics, Vol. XXXI (1909), p. 167.

§ 2. *Generalizations of the Tetrahedral Group.*

It is well known that the tetrahedral group may be defined by each of the following sets of three equations:

$$s_1^3 = 1,\ s_2^3 = 1,\ (s_1 s_2)^2 = 1;\quad s_1^2 = 1,\ s_2^3 = 1,\ (s_1 s_2)^3 = 1.$$

Instead of the first of these sets, we shall consider the more general but closely related set of two equations:

$$s_1^3 = s_2^3,\ (s_1 s_2)^2 = (s_2 s_1)^2.$$

By equating the second to unity there results a set of equations which were considered in the generalization mentioned above, while defining relations of the tetrahedral group are obtained when each of these two equations is made equal to unity. Hence these equations may be regarded as a double generalization of defining relations of the tetrahedral group.

From the equation $(s_1 s_2)^2 = (s_2 s_1)^2$ it is easy to derive the following four:

$$(s_1 s_2^2)^n = (s_2 s_1)^{2n} s_2^{-1} s_1^{-n} s_2,\quad (s_2 s_1^2)^n = (s_1 s_2)^{2n} s_1^{-1} s_2^{-n} s_1,$$
$$(s_1^2 s_2)^n = (s_2 s_1)^{2n} s_1 s_2^{-n} s_1^{-1},\quad (s_2^2 s_1)^n = (s_1 s_2)^{2n} s_2 s_1^{-n} s_2^{-1}.$$

These equations are useful to simplify some of the relations given below. The group $\{s_1, s_2\} = G$, generated by s_1, s_2, involves s_1^3 and $(s_1 s_2)^2$ invariantly, since each of these operators is commutative with both s_1 and s_2. As $s_1 s_2$ and $s_2 s_1$ have a common square, the product of one of these operators into the inverse of the other is transformed into its inverse by each of them.* This property will be used to find an upper limit for the orders of some of the operators in G, as follows:

$$s_1 s_2 . s_1^{-1} s_2^{-1} = s_1^{-6} s_1 . s_2 s_1^2 . s_2^2 = s_1^{-6} (s_1 s_2)^2 s_2^{-1} s_1 s_2^2 = s_1^{-6} (s_1 s_2)^2 s_2^{-1} . s_1 s_2 . s_2,$$
$$(s_1 s_2 s_1^{-1} s_2^{-1})^2 = s_1^{-12} (s_1 s_2)^4 s_2^{-1} (s_1 s_2)^2 s_2 = s_1^{-12} (s_1 s_2)^6.$$

As $s_1^{-12} (s_1 s_2)^6$ is both invariant under $s_1 s_2$ and is also transformed into its inverse by this operator, its order is either 2 or 1.

Having found an upper limit for the order of $s_1^{-12} (s_1 s_2)^6$, it is easy to prove that the order of

$$Q = \{s_1^{-6} (s_1 s_2)^2 s_2^{-1} . s_1 s_2 . s_2 = s_1^{-6} (s_2^{-1} s_1 s_2^2)^3,\ \ s_1^{-6} (s_2 s_1)^3,\ \ s_1^{-6} (s_1 s_2)^3\}$$

divides 8 and that Q is invariant under G. The three given generators of Q form a complete set of conjugates under s_2, and hence Q is invariant under s_2. These generators have a common square, since the square of one is invariant under G. The product of the last into the inverse of the second has been proved to be equal to the first. As it has also been proved that the order of the first divides 4, it results that the order of Q must divide 8. As the last two given generators of Q are transformed among themselves by s_1, Q is invariant under s_1 as well as under s_2; that is, Q is invariant under G. Moreover, Q involves the commutator subgroup of G, since it includes

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* *Archiv der Mathematik und Physik*, Vol. IX (1905), p. 6.

$$s_1^{-6}(s_2 s_1)^3 \cdot s_1^6 (s_1 s_2)^{-3} = s_2 s_1 s_2^{-1} s_1^{-1}.$$

Hence every subgroup of G which involves Q is invariant under G.

When s_1, s_2 are commutative, the second relation $(s_1 s_2)^2 = (s_2 s_1)^2$ does not give any additional information, and hence G is any abelian group which may be generated by two operators having a common cube. That is, G may be any cyclic group, or the direct product of such a group and the group of order 3, whenever s_1, s_2 are commutative. This disposes of the case when s_1, s_2 are commutative, and we shall therefore assume that these operators are non-commutative during the rest of the discussion relating to the equations

$$s_1^3 = s_2^3, \quad (s_1 s_2)^2 = (s_2 s_1)^2.$$

As s_1, s_2 are non-commutative, the order of Q cannot be less than 4, and hence Q is either the quaternion group or the four group as the order of $s_1^{-12}(s_1 s_2)^6$ is 2 or 1. Since $(s_1 s_2)^4 s_1^{-8}$ is of order 3, it results from the above that $\{Q, (s_1 s_2)^4 s_1^{-8}\}$ is the non-twelve group of order 24 or the tetrahedral group as the order of $s_1^{-12}(s_1 s_2)^6$ is 2 or 1. It is also evident that Q is actually the commutator subgroup of G. This proves the theorem: *If two non-commutative operators satisfy the two conditions* $s_1^3 = s_2^3$, $(s_1 s_2)^2 = (s_2 s_1)^2$, *they generate a group whose commutator subgroup is either the tetrahedral group or the non-twelve group of order* 24.

The abelian group $\{s_1^3, (s_1 s_2)^2\}$ and the commutator subgroup of G clearly generate G, and these two groups have only the identity in common when the commutator subgroup is the tetrahedral group, since this involves no invariant operator besides the identity. That is, when the commutator subgroup of G is the tetrahedral group, G is the direct product of $\{s_1^3, (s_1 s_2)^2\}$ and $\{Q, (s_1 s_2)^4 s_1^{-8}\}$. When the commutator subgroup of G is the non-twelve group of order 24, it has exactly two operators in common with $\{s_1^3, (s_1 s_2)^2\}$.

Although the conditions $s_1^3 = s_2^3$, $(s_1 s_2)^2 = (s_2 s_1)^2$ are not sufficient to limit the order of G, they restrict G to a very elementary category of solvable groups. All of these groups are isomorphic to the tetrahedral group with respect to an abelian invariant subgroup; viz., $\{s_1^3, (s_1s_2)^2\}$. In other words, the quotient group of G with respect to this abelian subgroup is the tetrahedral group. By assigning a finite value either for the order of s_1^3 or for the order of $(s_1s_2)^2$, there results a finite number of possible groups. The possible groups for one such case, viz., when the order of $(s_1s_2)^2$ is 1, were considered in the article mentioned above with the heading, "Generalization of the Groups of Genus Zero." By employing the results given above, it would be an easy matter to determine all the possible groups in other cases.

The second generalization of the tetrahedral group may be represented by the two equations

$$s_1^2 = s_2^3, \quad (s_1s_2)^3 = (s_2s_1)^3.$$

Defining relations of the tetrahedral group may evidently be obtained by making each member of these equations equal to unity. To obtain an upper limit for the orders of some of the operators in G we may proceed as follows:

$$s_1, \quad s_2^{-1}s_1s_2, \quad s_2^{-2}s_1s_2^2 = s_2s_1s_2^{-1}$$

are three operators of G, and they have a common square since s_1^2 is invariant under G. From

$$s_2^{-1}s_1s_2 \cdot s_2s_1^{-1}s_2^{-1} = (s_2^{-1}s_1^{-1})^2s_2^5 = (s_1s_2)^{-3}s_1^5$$

it results that $(s_1s_2)^{-6}s_1^{10}$ is both invariant under $s_2^{-1}s_1s_2$ and is also transformed into its inverse by this operator. Hence $(s_1s_2)^{-6}s_1^{10}$ is of order 2 or 1.

The group

$$\{(s_1s_2)^{-3}s_1^5,\ (s_1s_2)^{-3}s_2^{-1}s_1^5s_2,\ (s_1s_2)^{-3}s_2s_1^5s_2^{-1}\} = Q$$

is invariant under s_2 since it is generated by a complete set of conjugates under s_2, and it is also transformed into itself by s_1 because it is transformed into itself by each of the three operators $(s_1s_2)^{-3}s_1^5$, $(s_1s_2)^3$, s_1^4. The last two of these operators are evidently invariant under G. As $(s_1s_2)^{-3}s_1^5$ is transformed into its inverse by $s_2^{-1}s_1s_2$, it must also be transformed into its inverse by $(s_1s_2)^{-3}s_2^{-1}s_1^5s_2$. Moreover, the three given generators of Q have a common square, and it has been proved that the product of the second into the inverse of the third is equal to the first. Hence it results that Q is the quaternion group or the four group as the order of $(s_1s_2)^{-6}s_1^{10}$ is 2 or 1. It is also easy to see that Q is the commutator subgroup of G, since it includes

$$(s_1s_2)^3s_1^{-5} \cdot (s_1s_2)^{-3}s_2^{-1}s_1^5s_2 = s_1^{-1}s_2^{-1}s_1s_2 = (s_1s_2)^{-3}s_2s_1^5s_2^{-1}.$$

Since G is generated by Q, s_2, $(s_1s_2)^3$, it is isomorphic with the tetrahedral group with respect to the abelian subgroup $\{s_1^2, (s_1s_2)^3\}$. From what precedes there results the following theorem: *If two non-commutative operators satisfy the conditions* $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$, *they generate a group* G *whose commutator subgroup is either the quaternion group or the four group and whose central is* $\{s_1^2, (s_1 s_2)^3\}$. *The quotient group of* G *with respect to its central is the tetrahedral group.*

In the case considered above when $s_1^3 = s_2^3$, $(s_1 s_2)^2 = (s_2 s_1)^2$, it was possible to prove that G involves a non-invariant operator of order 3 and hence such a G includes the tetrahedral group or the non-twelve group of order 24. In the present case it is not possible to prove the existence of such subgroups. In fact, it is not difficult to see that s_1, s_2 may be so selected that the order of every non-invariant operator whose order is a power of 3 exceeds any finite number. If s_1, s_2 are any two operators which are commutative with t and satisfy the relations $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$, the operators $t^a s_1$, $t s_2$ will also satisfy these relations, provided $t^3 = t^{2a}$. By selecting t in such a way that its order is an arbitrary odd number divisible by 3, the given relations may be satisfied, and hence it is possible to select t so as to make the order of every operator of G which is not in the group generated by its central and its commutator subgroup exceed any finite number.

Suppose that the order of $s_1 s_2$ is $3m$, m being prime to 3. Since $(s_1 s_2)^{12} = s_1^{20}$, the order of s_1 must be prime to 3. As s_1 must be of even order when G is non-abelian, we may assume that the order of s_1 is $2k$ and hence the order of s_2 is $3k$. Hence s_2^k is a non-invariant operator of order 3 and $\{s_2^k, Q\}$ is either the tetrahedral group or the non-twelve group of order 24. This group is invariant under G, as it involves the commutator subgroup of G. It is evident that G is generated by the two groups $\{s_1^4, (s_1 s_2)^3\}$ and $\{s_2^k, Q\}$, and that these groups have at most two common operators. If $(s_1 s_2)^3$ is of odd order, the former of these groups must be of odd order, and hence G is the direct product of these groups. Hence the theorem: *If two non-commutative operators satisfy the relations* $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$ *and if* $(s_1 s_2)^3$ *is of an odd order prime to* 3, *then these operators generate a group which is the direct product of the cyclic group generated by* s_1^4 *and either the tetrahedral group or the non-twelve group of order* 24 *as the order of* $(s_1 s_2)^{-6} s_1^{10}$ *is* 1 *or* 2. This includes the known theorem as regards the groups generated by two operators satisfying the relations $s_1^2 = s_2^3$, $(s_1 s_2)^3 = 1$.*

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 4.

In what precedes, it was assumed that s_1, s_2 are not commutative. If they are commutative, the relation $(s_1 s_2)^3 = (s_2 s_1)^3$ does not imply any further restriction and hence G is cyclic, being generated by two commutative operators which satisfy the relation $s_1^\alpha = s_2^\beta$, where α and β are relatively prime. That is, if two commutative operators satisfy the relations $s_1^2 = s_2^3$, $(s_1 s_2)^3 = (s_2 s_1)^3$, they generate a cyclic group and every cyclic group may be generated by two such operators.

§ 3. *Generalizations of the Octahedral Group.*

By assuming that each member of the equations

$$s_1^2 = s_2^4, \quad (s_1 s_2)^3 = (s_2 s_1)^3$$

is equal to the identity, we arrive at well-known defining relations of the octahedral group. Hence these relations may be regarded as a double generalization of the octahedral group, and it is our first object to study the fundamental properties of the system of groups which may be generated by two non-commutative operators satisfying these general relations. As in the cases considered in the preceding section, we shall first find an upper limit for the orders of some operators contained in such a group G.

The three operators

$$s_2^2, \quad s_2^{-1} s_1^{-1} s_2^2 s_1 s_2, \quad s_1 s_2^2 s_1^{-1}$$

constitute a complete set of conjugates under $s_1 s_2$. As they have a common square, any two of them generate the generalized dihedral group. The product of the third into the inverse of the second is

$$\begin{aligned} s_1 s_2^2 s_1^{-1} s_2^{-1} s_1^{-1} s_2^{-2} s_1 s_2 &= s_1 s_2^2 (s_2 s_1)^{-2} s_2^{-1} s_1 s_2 = (s_1 s_2)^{-3} s_1 s_2^3 s_1 s_2^{-1} s_1 s_2 \\ &= (s_1 s_2)^{-3} (s_1 s_2^3)^3 s_2^{-6} = (s_1 s_2)^{-6} s_2^{18}. \end{aligned}$$

As $(s_1 s_2)^{-12} s_2^{36}$ is both invariant under G and also transformed into its inverse by $s_1 s_2^2 s_1^{-1}$, its order is 2 or 1, and $(s_1 s_2)^{24} = s_2^{72}$.

The group

$$\{(s_1 s_2)^{-6} s_2^{18}, \; s_2^{-1} s_1^{-1} s_2^{18} s_1 s_2 (s_1 s_2)^{-6}, \; s_1 s_2^{18} s_1^{-1} (s_1 s_2)^{-6}\} = Q$$

is invariant under $s_1 s_2$, since it is generated by a complete set of conjugates under this operator. It is also invariant under s_1, since s_1 transforms the first and last of its generators among themselves and these two operators generate Q, as the first has been proved equal to the product of the third into the inverse of the second. From the last fact it results also that Q is the quaternion group or the four group as the order of $(s_1 s_2)^{-12} s_2^{36}$ is 2 or 1. As Q is invariant under both $s_1 s_2$ and s_1, it is invariant under G.

From the fact that the non-invariant operator

$$s_2^{24} (s_2 s_1)^{-8}$$

is of order 3, it results that $\{Q, s_2^{24} (s_2 s_1)^{-8}\}$ is either the tetrahedral group or the non-twelve group of order 24. We proceed to prove that this is the commutator subgroup of G. In fact, this results from the following product of operators contained in $\{Q, s_2^{24} (s_2 s_1)^{-8}\}$:

$$s_2^{18} (s_2 s_1)^{-6} \cdot s_2^{24} (s_2 s_1)^{8} = s_2^{-6} (s_2 s_1)^2 = s_2^{-1} s_1^{-1} s_2 s_1 .$$

As $\{Q, s_2^{24} (s_2 s_1)^{-8}\}$ is the commutator subgroup of G, it results that every other subgroup of G which involves this subgroup is invariant under G. In particular, the commutator subgroup and $s_2^9 (s_1 s_2)^{-3}$ generate an invariant subgroup of G whose order is twice the order of this commutator subgroup. This invariant subgroup is the symmetric group of order 24 if $(s_1 s_2)^{-12} s_2^{36}$ is the identity. If this operator is of order 2, it is a known group of order 48 involving the non-twelve group of order 24. The central of G is evidently $\{s_1^2, (s_1 s_2)^3\}$ and G is generated by its central and the group $\{Q, s_2^{24} (s_2 s_1)^{-8}, s_2^9 (s_1 s_2)^{-3}\}$. Hence the theorem: *If two non-commutative operators satisfy the conditions* $s_1^2 = s_2^4$, $(s_1 s_2)^3 = (s_2 s_1)^3$, *they generate a group whose commutator subgroup is either the tetrahedral group or the non-twelve group of order* 24, *and whose central is generated by* s_1^2, $(s_1 s_2)^3$. *The quotient group with respect to this central is the symmetric group of order* 24, *and the entire group is generated by its central and a group whose order divides* 48.

As G is evidently generated by the abelian group $\{s_1^8, (s_1 s_2)^3\}$ and the group $\{Q, s_2^{24} (s_2 s_1)^{-8}, s_2^9 (s_1 s_2)^{-3}\}$ whose order divides 48, it results that G is the direct product of these groups whenever $s_1 s_2$ is of odd order. In this case $\{s_1^8, (s_1 s_2)^3\}$ is the cyclic group generated by s_1^8, and hence it is easy to determine all the possible groups in which $(s_1 s_2)^3$ has for its order a given odd number. The particular case when this order is unity has been considered, and the results constitute a special case of what has just been proved.*

A second generalization of the octahedral group is given by the equations

$$s_1^2 = s_2^3, \quad (s_1 s_2)^4 = (s_2 s_1)^4 .$$

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* *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 6.

In this case we shall consider the transforms of $(s_1 s_2)^2$ with respect to s_2. The complete set of these transforms is

$$(s_2 s_1)^2, \quad (s_1 s_2)^2, \quad (s_2^{-1} s_1 s_2^2)^2.$$

The product of the second into the inverse of the third is

$$\begin{aligned} s_1 s_2 s_1 s_2^{-1} s_1^{-1} s_2^{-1} s_1^{-1} s_2 &= s_1 s_2 s_1^2 (s_2 s_1)^{-3} s_2^2 = (s_1 s_2)^{-4} s_1 s_2 s_1^2 . s_2 s_1 . s_2^2 \\ &= (s_1 s_2)^{-4} s_1 s_2^2 s_1 s_2^2 s_1^2 = (s_2 s_1)^{-6} s_1^{10}. \end{aligned}$$

Since $(s_2 s_1)^{-12} s_1^{20}$ is both invariant under G and also transformed into its inverse by $(s_1 s_2)^2$, its order is 2 or 1.

It will now be proved that the group

$$\{(s_2 s_1)^{-6} s_1^{10}, \ (s_1 s_2)^{-6} s_1^{10}, \ (s_2^{-1} s_1 s_2^2)^{-6} s_1^{10}\} = Q$$

is invariant under G and that it is either the four group or the quaternion group. From the equations given above, it results directly that the product of the second generator of Q and the inverse of the third is the first, and hence Q is generated by any two of the given set. Since the first of these generators is transformed into its inverse by the second and since they have a common square, it results that Q is the quaternion group or the four group as $(s_1 s_2)^{-12} s_1^{20}$ is of order 2 or 1. As Q is generated by a complete set of conjugates under s_2, it is invariant under this operator. It is also invariant under s_1, since s_1 transforms the first two generators among themselves. This completes the proofs of the facts that Q is invariant under G and that it is either the quaternion group or the four group.

The operator $(s_2 s_1)^{-8} s_2^{20}$ is a non-invariant operator of order 3, and hence $\{Q, (s_2 s_1)^{-8} s_2^{20}\}$ is the tetrahedral group or the non-twelve group of order 24 as $(s_2 s_1)^{-12} s_2^{30}$ is of order 1 or 2. This is also the commutator subgroup (C) of G, since it includes

$$(s_1 s_2)^8 s_2^{-20} . (s_1 s_2)^6 s_2^{15} = (s_1 s_2)^2 s_2^{-5} = s_1 s_2 s_1^{-1} s_2^{-1}$$

and C can clearly not be a subgroup of $\{Q, (s_2 s_1)^{-8} s_2^{20}\}$. Having proved that $C = \{Q, (s_2 s_1)^{-8} s_2^{20}\}$, it results directly that this group is invariant under G. The quotient group of G with respect to its central $\{s_1^2, (s_1 s_2)^4\}$ is the symmetric group of order 24. In the preceding generalization of the octahedral group it was possible to prove that G involves a group of order 24 or of order 48, which has an $(\alpha, 1)$ isomorphism with this quotient group. In the present generalization, on the contrary, it is not possible to prove the existence of any finite group in G having a $(\beta, 1)$ isomorphism with the quotient group with respect to the central, since there is no limit to the highest power of 2 that may occur in the order of every operator of G not contained in the group generated by its central and its commutator subgroup.

It is evident that $\{s_1^6, (s_1 s_2)^8, (s_2 s_1)^{-8} s_2^{20}\} \equiv \{s_1^2, s_2\}$, and that this is the group generated by the central and the commutator subgroup of G. If it is assumed that s_1^2 is of odd order, it results that $s_1^{2k} = s_1^{40}$, k being an odd number, and hence $s_1^k (s_1 s_2)^{-12}$ is an operator of order 2 which is not contained in $\{s_1^2, s_2\}$. By adjoining $s_1^k (s_1 s_2)^{-12}$ to $\{Q, (s_2 s_1)^{-8} s_2^{20}\}$ there results a group which has an $(\alpha, 1)$ isomorphism with the octahedral group, where $\alpha = 2$ or 1. In this case the abelian group $\{s_1^6, (s_1 s_2)^8\}$ is clearly of odd order, and G is the direct product of this abelian group and the group $\{Q, (s_2 s_1)^{-8} s_2^{20}, s_1^k (s_1 s_2)^{-12}\}$. Hence the theorem: *If two non-commutative operators satisfy the equations* $s_1^2 = s_2^3$, $(s_1 s_2)^4 = (s_2 s_1)^4$, *they generate a group whose commutator subgroup is either the tetrahedral group or the non-twelve group of order* 24. *If* s_1^2 *is of odd order, the group* $\{s_1, s_2\}$ *is the direct product of an abelian group of odd order and a group whose order divides* 48.

The third generalization of the octahedral group is given by the equations

$$s_1^3 = s_2^4, \quad (s_1 s_2)^2 = (s_2 s_1)^2.$$

In the preceding section we deduced four useful formulas directly from the latter of these equations. These formulas will be used in what follows. Just as in the preceding two cases, we shall prove that the commutator subgroup of G is either the tetrahedral group or the non-twelve group of order 24. The steps towards this proof may be taken as follows:

$$s_2^{-2} \,.\, s_1^{-2} s_2^2 s_1^2 = s_2^{-8} (s_2^2 s_1)^2 s_1 = s_2^{-8} (s_1 s_2)^4 s_2 s_1^{-2} s_2^{-1} s_1 = s_2^{-12} (s_1 s_2)^6 s_1^{-1} s_2^{-2} s_1.$$

Hence $(s_2^{-2} s_1^{-2} s_2^2 s_1^2)^2 = s_2^{-28} (s_1 s_2)^{12}$. As this operator is both invariant under G and also transformed into its inverse by s_2^2, its order is 2 or 1.

The group

$$Q = \{s_2^{-12} (s_1 s_2)^6 s_1^{-1} s_2^{-2} s_1,\ s_2^{-12} (s_1 s_2)^6 s_1 s_2^{-2} s_1^{-1},\ s_2^{-12} (s_1 s_2)^6 s_2^{-2}\}$$

is generated by a complete set of conjugates under s_1, and the product of the third generator and the inverse of the second has been proved equal to the first, since $s_1 s_2^{-2} s_1^{-1} = s_1^{-2} s_2^{-2} s_1^2$. It is also evident that the second and the third of these generators transform the first into its inverse and that these three generators have a common square. Hence Q is the quaternion group or the four group as $s_2^{-28} (s_1 s_2)^{12}$ is of order 2 or 1. Since Q is invariant under s_1 and also under $s_2 s_1$, it must be invariant under G.

As $s_1^{-14}(s_1 s_2)^8$ is a non-invariant operator of order 3, the group $\{Q, s_1^{-14}(s_1 s_2)^8\}$ is either the tetrahedral group or the non-twelve group of order 24. It is also the commutator subgroup of G, since it includes

$$s_1^{-14}(s_1 s_2)^8 \cdot s_2^{14}(s_1 s_2)^{-6} = s_1^{-2}(s_1 s_2)^2 s_2^{-2} = s_1^{-1} s_2 s_1 s_2^{-1}$$

and cannot be larger than the commutator subgroup. The central of G $\{s_1^3, (s_1 s_2)^2\}$ and this commutator subgroup clearly generate $\{s_1, s_2^2\}$. As in the preceding case it is impossible to assign an upper limit for the minimum order of an operator of G which is not contained in this subgroup, since the condition $s_1^3 = s_2^4$ implies that such an order may be divisible by an arbitrary power of 2.

If we assumed that $(s_1 s_2)^2$ is of odd order, it results that $(s_1 s_2)^{2k} = (s_1 s_2)^{24}$, k being an odd number. Hence $s_2^{-28}(s_1 s_2)^k$ is an operator of order 2 which is not found in $\{s_1, s_2^2\}$. The group $\{Q, s_1^{-14}(s_1 s_2)^8, s_2^{-28}(s_1 s_2)^k\}$ is invariant, and its order is twice that of the commutator subgroup of G. This group and the abelian group of odd order $\{s_1^6, (s_1 s_2)^2\}$ must generate G; and as the latter is composed of operators of odd order, G is the direct product of these two groups. Hence the theorem: *If two non-commutative operators satisfy the equations* $s_1^3 = s_2^4$, $(s_1 s_2)^2 = (s_2 s_1)^2$, *and if the order of* $(s_1 s_2)^2$ *is odd, the group generated by these operators is the direct product of an abelian group of odd order* $\{s_1^6, (s_1 s_2)^2\}$ *and a group whose order divides* 48. *The latter has an* $(\alpha, 1)$ *isomorphism with the octahedral group, where* $\alpha = 1$ *or* 2. *Only a finite number of groups is possible whenever the order of one of the operators* s_1, s_2, $s_1 s_2$ *is given.*

In what precedes, it was assumed that s_1, s_2 are non-commutative. If these operators are commutative, each of these three cases becomes very simple since the second equations do not imply any further restriction. Hence it results that two commutative operators which satisfy separately the three pairs of equations

$$s_1^2 = s_2^4,\ (s_1 s_2)^3 = (s_2 s_1)^3; \quad s_1^2 = s_2^3,\ (s_1 s_2)^4 = (s_2 s_1)^4; \quad s_1^3 = s_2^4,\ (s_1 s_2)^2 = (s_2 s_1)^2$$

generate in the first case either any cyclic group or the direct product of such a group and the group of order 2, while in each of the other two cases they can generate only cyclic groups.